2 Distributions

Exercise 2.1. Show that p.v.$\dfrac{1}{x} \in \mathscr{S}'(\mathbb{R})$.

Exercise 2.2. Show that for all $\varphi \in \mathscr{D}(\mathbb{R})$, and for all $n \in \mathbb{N}$, there exists $C_n < \infty$ such that

$$\sup_{|\alpha|,|\beta| \leq n} \|\tau_a \varphi\|_{\alpha,\beta} \leq C_n (1+|a|)^n.$$

Deduce from this fact that $e^x \notin \mathscr{S}'(\mathbb{R})$.

Exercise 2.3. Show that the distribution

$$T = \sum_{n \in \mathbb{Z}} a_n \, \delta_n$$

belongs to $\mathscr{S}'(\mathbb{R})$ if and only if $\{a_n\}_{n \in \mathbb{N}}$ has polynomial growth, *i.e.*

$$|a_n| \leq C(1+|n|)^N$$

for some $C < \infty$ and $N \in \mathbb{N}$.

Exercise 2.4. Let $T \in \mathscr{E}'(\mathbb{R}^d)$ and $\varphi \in \mathscr{S}(\mathbb{R}^d)$. Show that $T * \varphi \in \mathscr{S}(\mathbb{R}^d)$, and that for all $\alpha, \beta \in \mathbb{N}^d$, there exists $N \in \mathbb{N}$, and $C_{\alpha,\beta} < \infty$ such that

$$\|T * \varphi\|_{\alpha,\beta} \leq C_{\alpha,\beta} \sup_{|\alpha'|,|\beta'| \leq N} \|\varphi\|_{\alpha',\beta'} .$$

Deduce that for all $S \in \mathscr{S}'(\mathbb{R}^d)$, we have $T * S \in \mathscr{S}'(\mathbb{R}^d)$.

Exercise 2.5. Compute the Fourier transform of $\dfrac{1}{|x|^2}$ in $\mathbb{R}^3$, and use this result to derive the fundamental solution of the Laplace equation.